$$P(y \mid \beta, a, \sigma, r_0) = \left(\frac{1}{(\sqrt{2\pi}\,\sigma)^n}\right)\left(\prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}}\right) \cdot e^{-\frac{(r_1 - a r_0)^2}{2\sigma^2}}$$

$$P(r_0 \mid \beta, a, \sigma) = N\left(0, \sigma^2 / (1-a^2)\right)$$

$$= \frac{1}{\sqrt{2\pi} \cdot \frac{\sigma}{\sqrt{1-a^2}}} e^{-r_0^2 / 2 \cdot \frac{\sigma^2}{(1-a^2)}}$$

$$P(y \mid \beta, a, \sigma) = \int P(y \mid \beta, a, \sigma, r_0)\, P(r_0 \mid \beta, a, \sigma)\, dr_0$$

$$= \frac{\sqrt{1-a^2}}{(\sqrt{2\pi} \cdot \sigma)^{n+1}} \cdot \prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}} \int e^{\frac{-(1-a^2) r_0^2 - (r_1 - a r_0)^2}{2\sigma^2}} dr_0$$

$$= \frac{\sqrt{1-a^2}}{(\sqrt{2\pi}\,\sigma)^{n+1}} \prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}} \cdot e^{-\frac{r_1^2}{2\sigma^2}} \int e^{\frac{-r_0^2 - 2a r_1 r_0}{2\sigma^2}} dr_0$$

$$= \frac{\sqrt{1-a^2}}{(\sqrt{2\pi}\,\sigma)^{n+1}} \prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}}\, e^{-\frac{r_1^2}{2\sigma^2}}\, e^{\frac{a^2 r_1^2}{2\sigma^2}} \int e^{\frac{-(r_0 + a r_1)^2}{2\sigma^2}} dr_0$$

$$= \frac{\sqrt{1-a^2}}{(\sqrt{2\pi}\,\sigma)^{n+1}} \cdot e^{-\frac{(1-a^2) r_1^2}{2\sigma^2}} \cdot \sigma \cdot \int e^{\frac{-(r_0 + a r_1)^2}{2\sigma^2}} d\left(\frac{r_0 + a r_1}{\sigma}\right) \cdot \prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}}$$

$$\bigstar \ \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du = 1}$$

$$= \frac{\sqrt{1-a^2}}{(\sqrt{2\pi}\,\sigma)^{n+1}} \cdot e^{-\frac{(1-a^2) r_1^2}{2\sigma^2}} \cdot \sigma \cdot \sqrt{2\pi} \cdot \prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}}$$

$$= \frac{\sqrt{1-a^2}}{(\sqrt{2\pi}\,\sigma)^{n}} e^{-\frac{(1-a^2) r_1^2}{2\sigma^2}} \prod_{i=2}^{n} e^{-\frac{(r_i - a r_{i-1})^2}{2\sigma^2}} \quad \triangle$$