General Linear Model with Correlated Errors

Assume that we have a series of n observations, $\mathbf{y} = (y_1, \ldots, y_n)$ and p regressors $x_1, \ldots, x_p$, where x_i is a row vector with n elements and $i = 1, \ldots, p$. We can model $\mathbf{y}$ as a function of the x_i's using the General Linear Model and write

$$\mathbf{y} = \mathbf{X\beta} + \varepsilon$$

where $\mathbf{X} = (x_1^t, \ldots, x_p^t)$ and $\boldsymbol{\beta}$ is a $p \times 1$ column vector of regression coefficients, to be estimated.

Assume that ε is a first-order autoregressive process ("AR(1)"). That is, $\varepsilon_i = a\varepsilon_{i-1} + n_i$, where a is the *autocorrelation parameter* and n_i is white noise, $\sim N(0, \sigma^2)$ Further assume that $|a| \leq 1$, so that ε is a stationary process. It can be shown that the following is the joint probability of the n observations:

$$P(\mathbf{y}|\boldsymbol{\beta}, a, \sigma, r_0) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^n} \prod_{i=1}^{n} e^{-\frac{(r_i - ar_{i-1})^2}{2\sigma^2}} \qquad [1]$$

where r_i are the residuals $r_i \equiv y_i - \mathbf{X}_i\boldsymbol{\beta}$, with $\mathbf{X}_i$ denoting the i^{th} row of $\mathbf{X}$. This density is conditioned on the residual at the unobserved time i = 0, r_0, which is a random variable with Gaussian distribution

$$P(r_0|\beta, a, \sigma) = N\left(0, \sigma^2 / (1 - a^2)\right) \qquad [2]$$

Since the time i = 0 is unobserved, it is desirable to integrate r_0 out to obtain a likelihood function that depends only on a, β, σ, and $r_1, r_2, \ldots, r_n$. It can be shown that doing so results in the following identity:

$$\begin{aligned} P(\mathbf{y}|\beta, a, \sigma) &= \int P(\mathbf{y}|\beta, a, \sigma, r_0) P(r_0|\beta, a, \sigma)\, dr_0 \\ &= \frac{\sqrt{1-a^2}}{\left(\sqrt{2\pi}\sigma\right)^n} e^{-\frac{(1-a^2)r_1^2}{2\sigma^2}} \prod_{i=2}^{n} e^{-\frac{(r_i - ar_{i-1})^2}{2\sigma^2}} \end{aligned} \qquad [3]$$

Problem: given [1] and [2], prove [3].